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Research on Fractional Exponential Function and Logarithmic Function

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***Abstract*: This paper introduces fractional exponential function and fractional logarithmic function. We obtain some properties of these two fractional functions. These properties are the same as those of classical exponential function and logarithmic function. The main method used in this paper is the chain rule for fractional derivatives based on Jumarie type of Riemann-Liouville (R-L) fractional calculus. A new multiplication and fractional analytic functions play important roles in this paper. The new multiplication is a natural operation in fractional calculus.**

***Keywords*: Fractional exponential function, Fractional logarithmic function, Some properties, Chain rule for fractional derivatives, Jumarie type of R-L fractional calculus, New multiplication, Fractional analytic functions.**

I. INTRODUCTION

In the second half of the 20th century, a large number of studies on fractional calculus were published in engineering literature. In fact, the latest progress of fractional calculus is mainly in differential and integral equations, physics, signal processing, hydrodynamics, viscoelasticity, mathematical biology and electrochemistry. There is no doubt that fractional calculus has become an exciting new mathematical tool to solve various problems in mathematics, science and engineering [1-9]. There is no unique definition of derivative and integral in fractional calculus. The commonly used definitions are Riemann-Liouville (R-L) fractional derivative, Caputo definition of fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative to avoid non-zero fractional derivative of constant function [10].

In this paper, we study the fractional exponential function and fractional logarithmic function. Based on Jumarie's modified R-L fractional calculus, some properties of these two fractional functions are obtained. A new multiplication, fractional analytic functions, and chain rule for fractional derivatives play important roles in this article. In fact, the new multiplication of fractional analytic functions is a generalization of the multiplication of traditional analytic functions.

II. PRELIMINARIES

First, the fractional calculus used in this paper is introduced below.

Definition 2.1 ([7])**:** If $0 < \alpha \leq 1$, and α is a real number. The Jumarie type of modified Riemann-Liouville α-fractional derivative and integral are defined respectively by

$$\left({}_{a}D_{\theta}^{\alpha}\right)[f(\theta)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\theta}\int_{a}^{\theta}\frac{f(t)-f(a)}{(\theta-t)^{\alpha}}dt, \tag{1}$$

$$\left({}_{a}I_{\theta}^{\alpha}\right)[f(\theta)] = \frac{1}{\Gamma(\alpha)}\int_{a}^{\theta}\frac{f(t)}{(\theta-t)^{1-\alpha}}dt, \tag{2}$$

where $\Gamma(\ )$ is the gamma function.

Proposition 2.2 ([11])**:** *Assume that* α, β, c *are real constants and* $\beta \geq \alpha > 0$, *then*

$$\left({}_0D_\theta^\alpha\right)[\theta^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\theta^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_0D_\theta^\alpha\right)[c] = 0. \tag{4}$$

Next, we define the fractional analytic function.

Definition 2.3 ([12])**:** Suppose that $, \theta_0$, and a_k are real numbers for all k, $\theta_0 \in (a,b)$, $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha}$ on some open interval $(\theta_0 - r, \theta_0 + r)$, then $f_\alpha(\theta^\alpha)$ is called α-fractional analytic at θ_0, where r is the radius of convergence about θ_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

The following is a new multiplication of fractional analytic functions.

Definition 2.4 ([12])**:** Assume that $0 < \alpha \leq 1$, $f_\alpha(\theta^\alpha)$ and $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions,

$$f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}\theta^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}, \tag{5}$$

$$g_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}\theta^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}. \tag{6}$$

We define

$$\begin{aligned} & f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}\theta^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}\theta^{k\alpha} \\ & = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)\theta^{k\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k} \\ & = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}. \end{aligned} \tag{8}$$

Definition 2.5: Let n be a positive integer, then $\left(f_\alpha(\theta^\alpha)\right)^{\otimes n} = f_\alpha(\theta^\alpha) \otimes \cdots \otimes f_\alpha(\theta^\alpha)$ is called the nth power of the fractional function $f_\alpha(\theta^\alpha)$. And $\left(f_\alpha(\theta^\alpha)\right)^{\otimes -n} = \left(f_\alpha(\theta^\alpha)\right)^{\otimes -1} \otimes \cdots \otimes \left(f_\alpha(\theta^\alpha)\right)^{\otimes -1}$ is the nth power of the fractional function $\left(f_\alpha(\theta^\alpha)\right)^{\otimes -1}$. In addition, we define $\left(f_\alpha(\theta^\alpha)\right)^{\otimes 0} = 1$. On the other hand, if $f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha) = 1$, then $g_\alpha(\theta^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(\theta^\alpha)$, and is denoted by $\left(f_\alpha(\theta^\alpha)\right)^{\otimes -1}$.

Remark 2.6: The $\otimes$ multiplication satisfies the commutative law and the associate law, and it is the generalization of ordinary multiplication, since the $\otimes$ multiplication becomes the traditional multiplication if $\alpha = 1$.

Definition 2.7: Let $0 < \alpha \leq 1$, and $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ be α-fractional analytic functions,

$$f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}\theta^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}, \tag{9}$$

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$$g_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} \theta^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} \theta^\alpha\right)^{\otimes k}. \quad (10)$$

Then we define the compositions of $f_\alpha(\theta^\alpha)$ and $g_\alpha(\theta^\alpha)$ are

$$(f_\alpha \circ g_\alpha)(\theta^\alpha) = f_\alpha\big(g_\alpha(\theta^\alpha)\big) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \big(g_\alpha(\theta^\alpha)\big)^{\otimes k}, \quad (11)$$

and

$$(g_\alpha \circ f_\alpha)(\theta^\alpha) = \ g_\alpha\big(f_\alpha(\theta^\alpha)\big) = \sum_{k=0}^{\infty} \frac{b_k}{k!} \big(f_\alpha(\theta^\alpha)\big)^{\otimes k}. \quad (12)$$

Definition 2.8: Let $0 < \alpha \leq 1$. If $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions such that

$$(f_\alpha \circ g_\alpha)(\theta^\alpha) = (g_\alpha \circ f_\alpha)(\theta^\alpha) = \frac{1}{\Gamma(\alpha+1)} \theta^\alpha. \quad (13)$$

Then we say that these two fractional analytic functions are inverse to each other.

In the following, we introduce the fractional exponential function and fractional logarithmic function.

Definition 2.9: The α-fractional exponential function is defined by

$$E_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{\theta^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{\Gamma(\alpha+1)} \theta^\alpha\right)^{\otimes k}. \quad (14)$$

And the α-fractional logarithmic function $Ln_\alpha(\theta^\alpha)$ is the inverse function of the $E_\alpha(\theta^\alpha)$. Where $0 < \alpha \leq 1$, and θ is a real variable.

Proposition 2.10: *If* $0 < \alpha \leq 1$ *and* θ, φ *are two real variables. Then*

$$E_\alpha(\theta^\alpha) \otimes E_\alpha(\varphi^\alpha) = E_\alpha(\theta^\alpha + \varphi^\alpha). \quad (15)$$

III. METHODS AND RESULTS

In this section, the methods and results are present. First, we introduce the main method used in this paper.

Theorem 3.1 (chain rule for fractional derivatives): *If* $0 < \alpha \leq 1$, *and let* $f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} \theta^\alpha\right)^{\otimes k}$, $g_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} \theta^\alpha\right)^{\otimes k}$. *Then*

$$\left({}_0D_\theta^\alpha\right)\left[f_\alpha\big(g_\alpha(\theta^\alpha)\big)\right] = \left({}_0D_\theta^\alpha\right)[f_\alpha(\theta^\alpha)]\big(g_\alpha(\theta^\alpha)\big) \otimes \left({}_0D_\theta^\alpha\right)[g_\alpha(\theta^\alpha)]. \quad (16)$$

Proof

$$\left({}_0D_\theta^\alpha\right)\left[f_\alpha\big(g_\alpha(\theta^\alpha)\big)\right]$$

$$= \left({}_0D_\theta^\alpha\right)\left[\sum_{k=0}^{\infty} \frac{a_k}{k!} \big(g_\alpha(\theta^\alpha)\big)^{\otimes k}\right]$$

$$= \sum_{k=0}^{\infty} \frac{a_k}{k!} \left({}_0D_\theta^\alpha\right)\left[\big(g_\alpha(\theta^\alpha)\big)^{\otimes k}\right]$$

$$= \sum_{k=1}^{\infty} \frac{a_k}{(k-1)!} \big(g_\alpha(\theta^\alpha)\big)^{\otimes (k-1)} \otimes \left({}_0D_\theta^\alpha\right)[g_\alpha(\theta^\alpha)]$$

$$= \left({}_0D_\theta^\alpha\right)[f_\alpha(\theta^\alpha)]\big(g_\alpha(\theta^\alpha)\big) \otimes \left({}_0D_\theta^\alpha\right)[g_\alpha(\theta^\alpha)].$$

Q.e.d

Theorem 3.2 : *Assume that* $0 < \alpha \leq 1$, *and* $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ *are two* α*-fractional analytic functions. Then*

$$E_\alpha\big(f_\alpha(\theta^\alpha)\big) \otimes E_\alpha\big(g_\alpha(\theta^\alpha)\big) = E_\alpha\big(f_\alpha(\theta^\alpha) + g_\alpha(\theta^\alpha)\big). \quad (17)$$

Proof

$$E_\alpha\big(f_\alpha(\theta^\alpha) + g_\alpha(\theta^\alpha)\big)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \big(f_\alpha(\theta^\alpha) + g_\alpha(\theta^\alpha)\big)^{\otimes k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{m=0}^{k} \binom{k}{m} \big(f_\alpha(\theta^\alpha)\big)^{\otimes (k-m)} \otimes \big(g_\alpha(\theta^\alpha)\big)^{\otimes m}$$

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$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{m=0}^{k} \binom{k}{m} \left(f_\alpha(\theta^\alpha) \right)^{\otimes(k-m)} \otimes \left(g_\alpha(\theta^\alpha) \right)^{\otimes m}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(f_\alpha(\theta^\alpha) \right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{1}{k!} \left(g_\alpha(\theta^\alpha) \right)^{\otimes k}$$

$$= E_\alpha\left(f_\alpha(\theta^\alpha) \right) \otimes E_\alpha\left(g_\alpha(\theta^\alpha) \right).$$ Q.e.d.

Theorem 3.3: *Let* $0 < \alpha \leq 1$ *, and* $f_\alpha(\theta^\alpha)$ *be an* α*-fractional analytic function. Then*

$$E_\alpha\left(- f_\alpha(\theta^\alpha)\right) = \left[E_\alpha\left(f_\alpha(\theta^\alpha)\right)\right]^{\otimes -1}. \tag{18}$$

Proof Since by Theorem 3.2,

$$1 = E_\alpha\left(f_\alpha(\theta^\alpha) - f_\alpha(\theta^\alpha)\right) = E_\alpha\left(f_\alpha(\theta^\alpha)\right) \otimes E_\alpha\left(- f_\alpha(\theta^\alpha)\right). \tag{19}$$

It follows that

$$E_\alpha\left(- f_\alpha(\theta^\alpha)\right) = \left[E_\alpha\left(f_\alpha(\theta^\alpha)\right)\right]^{\otimes -1}.$$ Q.e.d.

Theorem 3.4 : *Let* $0 < \alpha \leq 1$ *, n be an integer, and* $f_\alpha(\theta^\alpha)$ *be an* α*-fractional analytic function. Then*

$$E_\alpha\left(n\, f_\alpha(\theta^\alpha)\right) = \left[E_\alpha\left(f_\alpha(\theta^\alpha)\right)\right]^{\otimes n}. \tag{20}$$

Proof Using Theorem 3.2, Theorem 3.3, and by induction, we can easily obtain the desired result. Q.e.d.

Theorem 3.5: *Let* $0 < \alpha \leq 1$*, then*

$$Ln_\alpha(\theta^\alpha) = \left({}_1I_\theta^\alpha \right)\left[\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes -1}\right]. \tag{21}$$

Proof Since $E_\alpha\left(Ln_\alpha(\theta^\alpha)\right) = \frac{1}{\Gamma(\alpha+1)}\theta^\alpha$, by chain rule for fractional derivatives,

$$\left({}_0D_\theta^\alpha \right)\left[E_\alpha\left(Ln_\alpha(\theta^\alpha)\right)\right] = \left({}_0D_\theta^\alpha \right)\left[E_\alpha(\theta^\alpha)\right]\left(Ln_\alpha(\theta^\alpha)\right) \otimes \left({}_0D_\theta^\alpha \right)\left[Ln_\alpha(\theta^\alpha)\right]$$

$$= E_\alpha\left(Ln_\alpha(\theta^\alpha)\right) \otimes \left({}_0D_\theta^\alpha \right)\left[Ln_\alpha(\theta^\alpha)\right]$$

$$= \frac{1}{\Gamma(\alpha+1)}\theta^\alpha \otimes \left({}_0D_\theta^\alpha \right)\left[Ln_\alpha(\theta^\alpha)\right]. \tag{22}$$

On the other hand,

$$\left({}_0D_\theta^\alpha \right)\left[E_\alpha\left(Ln_\alpha(\theta^\alpha)\right)\right] = \left({}_0D_\theta^\alpha \right)\left[\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right] = 1. \tag{23}$$

And hence,

$$\left({}_0D_\theta^\alpha \right)\left[Ln_\alpha(\theta^\alpha)\right] = \left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes -1}. \tag{24}$$

Therefore,

$$Ln_\alpha(\theta^\alpha) = \left({}_1I_\theta^\alpha \right)\left[\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes -1}\right].$$ Q.e.d.

Theorem 3.6: *If* $0 < \alpha \leq 1$ *, and* $f_\alpha(\theta^\alpha), g_\alpha(\theta^\alpha)$ *are two* α*-fractional analytic functions. Then*

$$Ln_\alpha\left(f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha)\right) = Ln_\alpha\left(f_\alpha(\theta^\alpha)\right) + Ln_\alpha\left(g_\alpha(\theta^\alpha)\right). \tag{25}$$

Proof Since

$$E_\alpha\left(Ln_\alpha\left(f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha)\right)\right) = f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha). \tag{26}$$

And

$$E_\alpha\left(Ln_\alpha(f_\alpha(\theta^\alpha)) + Ln_\alpha(g_\alpha(\theta^\alpha))\right)$$
$$= E_\alpha\left(Ln_\alpha(f_\alpha(\theta^\alpha))\right) \otimes E_\alpha\left(Ln_\alpha(g_\alpha(\theta^\alpha))\right)$$
$$= f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha). \quad (27)$$

It follows that the desired result holds. Q.e.d.

Theorem 3.7: *If* $0 < \alpha \le 1$, *and* $f_\alpha(\theta^\alpha)$ *is an* α*-fractional analytic function. Then*

$$Ln_\alpha\left(\left(f_\alpha(\theta^\alpha)\right)^{\otimes -1}\right) = -Ln_\alpha(f_\alpha(\theta^\alpha)). \quad (28)$$

Proof Since by Theorem 3.6,

$$0 = Ln_\alpha\left(f_\alpha(\theta^\alpha) \otimes \left(f_\alpha(\theta^\alpha)\right)^{\otimes -1}\right) = Ln_\alpha(f_\alpha(\theta^\alpha)) + Ln_\alpha\left(\left(f_\alpha(\theta^\alpha)\right)^{\otimes -1}\right). \quad (29)$$

It follows that

$$Ln_\alpha\left(\left(f_\alpha(\theta^\alpha)\right)^{\otimes -1}\right) = -Ln_\alpha(f_\alpha(\theta^\alpha)).$$ Q.e.d.

Theorem 3.8: *If* $0 < \alpha \le 1$, n *is an integer, and* $f_\alpha(\theta^\alpha)$ *is an* α*-fractional analytic function. Then*

$$Ln_\alpha\left(\left(f_\alpha(\theta^\alpha)\right)^{\otimes n}\right) = n \cdot Ln_\alpha(f_\alpha(\theta^\alpha)). \quad (30)$$

Proof Using Theorem 3.6, Theorem 3.7, and by induction yields the desired result holds. Q.e.d.

IV. CONCLUSION

This paper discusses several properties of fractional exponential function and logarithmic function. These properties are the same as those of traditional exponential function and logarithmic function. The main method we used is the chain rule for fractional derivatives based on Jumarie's modified R-L fractional calculus. A new multiplication and fractional analytic functions play vital roles in this article. In fact, the new multiplication we defined is a natural operation in fractional calculus. In the future, we will use fractional exponential function and logarithmic function to study applied mathematics and fractional calculus.

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